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Regular C^1-Parametrizations for Exponential Sums and Splines

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1. INTRODUCTION

Approximation problems are commonly dealt with by introducing a suitable parametrization of the family of approximating functions under consideration. Thus, Werner and Braess' description of exponential sums via difference quotients [2] remains homeomorphic even for coalescing frequencies and therefore greatly simplifies all problems related to the topology of that manifold (e.g., existence of best approximations).

For other problems, though (like deriving characterizations of (local) best approximations or designing algorithms to calculate good approximations), information is needed not only on the *topological* but also on the *differential structure* of the manifold. The above-mentioned parametrization fails to supply that information when frequencies coalesce.

We therefore present in this paper a parametrization of γ-polynomials (e.g., exponential sums and splines) which retains its *differentiability and regularity* properties even for coalescing frequencies. This helps to derive necessary and sufficient conditions for local best approximations in a geometric, illustrative way [5], which in turn is a prerequisite for developing numerical procedures for the calculation of approximations [6]. One purpose of this paper is to point out that the geometry of such manifolds as exponential sums and splines have many features in common—in the spirit of the investigations by Hobby and Rice [8] and de Boor [1].

Major difficulties in designing such regular C^1-parametrizations stem from the fact that the tangent cones of γ-polynomials suffer from a loss of dimension when frequencies coalesce [1].

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In Section 2 the parametrization is defined and basic properties are derived. In Section 3 the differentiability and regularity of the parametrization is proved. Since some of the formulas for nth degree γ-polynomials may look a bit complicated at first sight, we have included in Section 4 a short discussion of the situation for γ-polynomials of degree 3. The tangent cones are explicitly calculated for this special case and our parametrization is compared to classical parametrizations. The reader who is not already motivated for the technicalities of Sections 2 and 3 is *strongly urged to read Section* 4 *before or parallel to studying the general case.* In a subsequent paper [5] the analysis of this article is used to derive necessary and sufficient conditions for γ-polynomials to be (local) best approximations.

The following notations will be used throughout this paper: Let n be a natural number, $T \subset \mathbb{R}$ with $\mathring{T} \supset T$, $X \subset \mathbb{R}$ with X compact, not necessarily finite, but $|X| \geqslant 2n+1$. For given $\gamma \in C(T \times X)$, the set of continuous real-valued functions on $T \times X$, the difference quotient $\Delta_t^j(t_1,..., t_k)\,\gamma(t, x)$ of γ with respect to the argument t is defined as usual [7]. If the partial derivatives of γ exist up to order $n-1$ and are continuous with respect to t and x, we can define the set of γ-polynomials of degree n:

$$\Gamma_n^\gamma := \left\{ \sum_{i=1}^{n} a_i \Delta_t^{i-1}(t_1,..., t_i)\,\gamma(t, \cdot) \,|\, a_i, t_i \in \mathbb{R};\, t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \right\}.$$

By l we denote the number of *distinct* characteristic numbers t_i and $m_1,..., m_l$ are their respective *multiplicities* ($\sum_{i=1}^{l} m_i = n$). Possible kernels γ are: e^{tx} (exponential sums), $(x-t)_+^m$ (splines), x^t, $\cosh(tx)$, $\operatorname{arctg}(tx)$, $(1+tx)^{-1}$ (related to rational functions), $(x-t)^{2n-1}$ (polynomial γ-polynomials), and others [3, 8]. A kernel $\gamma \in C(T \times X)$ is said to be an *m-kernel*, if (1) all partial derivatives of γ with respect to t exist up to order m and are continuous in both arguments, and if (2) in addition, for all $l \in \mathbb{N}$, $t_1 < t_2 < \cdots < t_l \in T$, the $l \cdot (m+1) \leqslant |X| - 1$ functions

$$\frac{\partial^i}{\partial t^i}\gamma(t, \cdot)|_{t=t_j}, \qquad 0 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant l,$$

are linearly independent as elements of $C(X)$. For an $(m-1)$-kernel γ ($m \leqslant n$) the set $\Gamma_{n,m}^\gamma$ is well defined by

$$\Gamma_{n,m}^\gamma := \left\{ \sum_{i=1}^{l} \sum_{j=1}^{m_i} a_{ij} \Delta_t^{j-1}(t_1,..., t_i)\,\gamma(t, \cdot) \,|\, l, m_i \in \mathbb{N}, m_i \leqslant m, \right.$$
$$\left. \sum_{i=1}^{l} m_i = n, a_{ij} \in \mathbb{R}, t_1 < t_2 < \cdots < t_l \right\}.$$

We have $\Gamma_{n,n}^\gamma = \Gamma_n^\gamma$.

γ is a *normal* m-kernel if γ is an m-kernel and for each $g \in \Gamma_{n,k}^{\gamma}$ $(n \geqslant k \leqslant m+1)$ there is a $\|\cdot\|_{\infty}$-neighborhood U of g and a compact subset of T such that the characteristic numbers $t_1, \ldots, t_n$ of all elements of $(\Gamma_{n,k}^{\gamma} \backslash \Gamma_{n-1,k}^{\gamma}) \cap U$ belong to the compact subset of T.

For a normal $(n-1)$-kernel γ the above parametrization induces a homeomorphism from $\Gamma_n^{\gamma} \backslash \Gamma_{n-1}^{\gamma}$ to the corresponding subset of $\mathbb{R}^{2n}$: The manifold cannot bend back and cross itself; see Braess [3, p. 37].

For a given Lebesgue-measure on X the induced norm on $C(X)$ is denoted by $\|\cdot\|_q$ for $1 \leqslant q \leqslant \infty$.

To *simplify formulas* let us agree to interpret $\sum_{i=j}^{k} a_i$ as *zero* for $k<j$, to interpret $i \cdot (\cdots)$ as *zero* for $i=0$ even if $(\cdots)$ is not explicitly defined, and to interpret $\binom{k}{i}$ as *zero* for $i<0$ or $i>k$.

2. A Parametrization of Γ_n^{γ}

Our parametrization is designed to describe the neighborhood of a given γ-polynomial g. Since the neighborhood depends on g (for example, the dimension of the tangent space or tangent cone varies with g), our parametrization also depends on g.

Choose natural numbers $n, l, m_1, \ldots, m_l$ with $\sum_{i=1}^{l} m_i = n$. These will be held fixed throughout this paper.

We set $T := \mathbb{R}$; the following analysis can easily be extended to cases where T is not the whole line, but rather a subset such as an open, closed, or half-closed interval.

Our parametrization distinguishes linear and nonlinear parameters. The nonlinear parameters are given by

$$\begin{aligned} M := \Big\{ & (\tau_1, \delta_1^{(1)}, \delta_2^{(1)}, \ldots, \delta_{m_1-1}^{(1)}, \tau_2, \delta_1^{(2)}, \ldots, \tau_l, \delta_1^{(l)}, \ldots, \delta_{m_l-1}^{(l)})^T \in \mathbb{R}^n \,\Big| \\ & 0 \leqslant \delta_i^{(k)} \leqslant \left(\frac{i+2}{i}\right)^2 \delta_{i+1}^{(k)}; \\ & \tau_j + (m_j - 1)\sqrt{\delta_{m_j-1}^{(j)}} \leqslant \tau_{j+1} - \sum_{q=1}^{m_{j+1}-1} \sqrt{\delta_q^{(j+1)}}, \\ & 1 \leqslant k \leqslant l;\ 1 \leqslant i \leqslant m_k - 2,\ 1 \leqslant j \leqslant l-1 \Big\}. \end{aligned}$$

For $l = n$ this simplifies to

$$M := \{(\tau_1, \tau_2, \ldots, \tau_n)^T \in \mathbb{R}^n \mid \tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_n\}$$

and the special case $l = 1$ yields

$$M := \left\{ (\tau_1, \delta_1^{(1)}, \delta_2^{(1)}, \ldots, \delta_{n-1}^{(1)})^T \in \mathbb{R}^n \,|\, 0 \leqslant \delta_i^{(1)} \leqslant \left(\frac{i+2}{i}\right)^2 \delta_{i+1}^{(1)}; \right.$$

$$\left. 1 \leqslant i \leqslant n-2 \right\}.$$

For a given $(n-1)$-kernel γ the parametrization $p: \mathbb{R}^n \times M \to \Gamma_n^\gamma$ is well defined by

$$\mathbb{R}^n \times M \ni (a_1, \ldots, a_n, \tau_1, \delta_1^{(1)}, \ldots, \delta_{m_l-1}^{(l)})^T \xrightarrow{p}$$

$$\sum_{i=1}^{l} \sum_{j=1}^{m_i} a_{s_i+j} \sum_{1+s_i \leqslant i_1 < i_2 < \cdots < i_j \leqslant s_{i+1}}$$

$$\Delta_t^{s_i+j-1}(t_1^{(1)}, \ldots, t_{m_1}^{(1)}, t_1^{(2)}, \ldots, t_{m_i}^{(i-1)}, t_{i_1}^{(i)}, t_{i_2}^{(i)}, \ldots, t_{i_j}^{(i)})\, \gamma(t, \cdot),$$

where for short the following notations were used:

$$s_j := \sum_{k=1}^{j-1} m_k, \quad 1 \leqslant j \leqslant l+1,$$

$$t_{i+1}^{(j)} := \tau_j + i\sqrt{\delta_i^{(j)}} - \sum_{k=i+1}^{m_j-1} \sqrt{\delta_k^{(j)}}, \quad 1 \leqslant j \leqslant l, 0 \leqslant i \leqslant m_j - 1.$$

Again, this simplifies in special situations. For $l = n$ we have

$$\mathbb{R}^n \times M \ni (a_1, \ldots, a_n, \tau_1, \ldots, \tau_n)^T \xrightarrow{p} \sum_{i=1}^{n} a_i \Delta_t^{i-1}(\tau_1, \ldots, \tau_i)\, \gamma(t, \cdot),$$

and for $l = 1$ we get

$$\mathbb{R}^n \times M \ni (a_1, \ldots, \delta_{n-1}^{(1)})^T \xrightarrow{p}$$

$$\sum_{j=1}^{n} a_j \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}^{(1)}, \ldots, t_{i_j}^{(1)})\, \gamma(t, \cdot)$$

with the $t_k^{(1)}$ as pictured in Table I where the parameter $i (1 \leqslant i \leqslant l)$ has been omitted. In the general case $(l > 1)$ we have l such groups of characteristic numbers.

The following theorem asserts that p indeed parametrizes the γ-polynomials of degree n.

THEOREM 1. *For an $(n-1)$-kernel $\gamma \in C(T \times X)$ p is well defined and parametrizes the set of γ-polynomials of degree n: $p(\mathbb{R}^n \times M) = \Gamma^\gamma{}_n$.*

TABLE I

The Characteristic Numbers for $l = 1$; the Index i $(1 \leqslant i \leqslant l)$ is Omitted

$$\begin{array}{lllll}
t_1 := \tau - \sqrt{\delta_1} - & \sqrt{\delta_2} - & \sqrt{\delta_3} - \sqrt{\delta_4} \cdots - \sqrt{\delta_{n-1}} \\
t_2 := \tau + \sqrt{\delta_1} - & \sqrt{\delta_2} - & \sqrt{\delta_3} - \sqrt{\delta_4} \cdots - \sqrt{\delta_{n-1}} \\
t_3 := \tau & + 2\sqrt{\delta_2} - & \sqrt{\delta_3} - \sqrt{\delta_4} \cdots - \sqrt{\delta_{n-1}} \\
t_4 := \tau & & + 3\sqrt{\delta_3} - \sqrt{\delta_4} \cdots - \sqrt{\delta_{n-1}} \\
\vdots & & \qquad\qquad \vdots \\
t_n := \tau & & \qquad + (n-1)\sqrt{\delta_{n-1}}
\end{array}$$

Proof. To avoid getting lost in technical details we give a proof of the above theorem only for $l = 1$. The proof can be extended to the general case without major difficulties. Since $l = 1$ we can omit the index ranging from 1 to l.

We note that the function defined in Table I induces a one-to-one correspondence between M and

$$\{(t_1, t_2, ..., t_n)^T \in \mathbb{R}^n \mid t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n\}.$$

To prove the theorem it is therefore sufficient to show that for arbitrary $t_i \in T$, $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, there exists a real lower triangular $n \times n$-matrix $L = (l_{ij})$ such that for all $a = (a_1, ..., a_n) \in \mathbb{R}^n$ we have, with $(b_1, ..., b_n) := La$,

$$\sum_{j=1}^{n} a_j \sum_{1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}, ..., t_{i_j})\, \gamma(t, \cdot)$$
$$= \sum_{i=1}^{n} b_i \Delta_t^{i-1}(t_1, ..., t_i)\, \gamma(t, \cdot)$$

and such that $\det(L) \neq 0$.

To prove this, we transform the original expression. There exist $a_{ij} \in \mathbb{R}$ depending continuously on $t_1, ..., t_n, a_1$ and *linear* in a_1 with

$$\sum_{j=1}^{n} a_j \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}, ..., t_{i_j})\, \gamma(t, \cdot)$$
$$= \binom{n}{1} a_1 \Delta_t^0(t_1)\, \gamma(t, \cdot)$$
$$+ \sum_{1 \leqslant i_1 < i_2 \leqslant n} (a_2 + a_{i_1 i_2})\, \Delta_t^1(t_{i_1}, t_{i_2})\, \gamma(t, \cdot)$$
$$+ \sum_{j=3}^{n} a_j \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}, ..., t_{i_j})\, \gamma(t, \cdot).$$

Only the recurrence formula $\Delta_t^0(t_i) - \Delta_t^0(t_j) = (t_i - t_j)\,\Delta_t^1(t_j, t_i)$ for difference quotients has been used here. With suitable $a_{ijk} \in \mathbb{R}^n$ each depending *either* on $t_1, t_2, \ldots, t_n$, a_1 and *linear* in a_1 *or* on $t_1, t_2, \ldots, t_n$, a_2 and *linear* in a_2 (not both) we continue this transformation to get

$$= \binom{n}{1} a_1 \Delta_t^0(t_1)\,\gamma(t, \cdot)$$

$$+ \left[\binom{n}{2} a_2 + \sum_{1 \leqslant i_1 < i_2 \leqslant n} a_{i_1 i_2}\right] \Delta_t^1(t_1, t_2)\,\gamma(t, \cdot)$$

$$+ \sum_{1 \leqslant i_1 < i_2 < i_3 \leqslant n} (a_3 + a_{i_1 i_2 i_3})\,\Delta_t^2(t_{i_1}, t_{i_2}, t_{i_3})\,\gamma(t, \cdot)$$

$$+ \sum_{j=4}^{n} a_i \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}, \ldots, t_{i_j})\,\gamma(t, \cdot).$$

By successive application of this process the elements of the matrix L can be constructed. For our purposes it is sufficient to conclude that $l_{ij} = 0$ for $1 \leqslant i < j \leqslant n$ and that $l_{ii} = \binom{n}{i}$, $1 \leqslant i \leqslant n$, which proves that the determinant of L is not zero. L is diagonal for $t_1 = t_2 = \cdots = t_n$. The reader should be aware of the fact that

$$g(\cdot) := \sum_{j=1}^{n} a_j \sum_{1 \leqslant i_1 < i_2 < \cdots \leqslant i_j \leqslant n} \Delta_t^{j-1}(t_{i_1}, \ldots, t_{i_j})\,\gamma(t, \cdot)$$

may be in Γ_{n-1}^{γ} although $a_n \neq 0$. But since L is diagonal with only positive entries for $t_1 = \cdots = t_n$ and because L depends continuously on $t_1, \ldots, t_n$ the γ-polynomial g is an element of $\Gamma_n^{\gamma} \backslash \Gamma_{n-1}^{\gamma}$ for sufficiently small $t_n - t_1$ only if $a_n \neq 0$. ∎

3. Differentiability and Regularity of p

In this section we discuss differentiability and regularity properties of p. At boundary points (at least two $t_i^{(j)}$ coalescing) p is not Fréchet-differentiable. Therefore we need a slightly modified definition of differentiability which can be obtained by relating consequently everything to the region of interest (domain). Note that the domain A of f need not be open nor do we require $\bar{A} \supset A$.

Definition 1. Let B_1 and B_2 be Banach-spaces, A a subset of B_1. A function $f: A \to B_2$ is said to be *differentiable* at $x \in A$ if there exists a

$f'(x) \in \mathscr{L}(B_1, B_2)$ (the set of continuous linear mappings from B_1 to B_2) such that for all $\varepsilon > 0$ there exists a $\delta > 0$ with

$$\|f(x+h) - f(x) - f'(x)(h)\| \leqslant \varepsilon \cdot \|h\|, \qquad \forall h \in B_1 : \|h\| \leqslant \delta, x+h \in A. \quad (*)$$

The derivative of f at $x \in A$ is said to be *singular* if

$$\varliminf_{\delta \to 0} \left(\inf_{\substack{x+h \in A \\ 0 \neq \|h\| < \delta}} \frac{1}{\|h\|} \|f'(x)(h)\| \right) = 0$$

and *regular* otherwise.

F is *continuously differentiable* at $z \in A$ ("f is C^1 at $z \in A$") if for a suitable B_1-neighborhood U of z there exists a map $f' : A \cap U \to \mathscr{L}(B_1, B_2)$ such that $(*)$ holds for all $x \in A \cap U$ and such that, in addition,

$$\varlimsup_{\delta \to 0} \sup_{\substack{z+h \in A \\ 0 \neq \|h\| < \delta}} \left(\frac{1}{\|h\|} \|f'(z)(h) - f'(x)(h)\| \right) \to 0, \quad \text{for } \|x - z\| \to 0. \quad \blacksquare$$

For open sets A the above differentiability and Fréchet-differentiability are the same. To illustrate the difference let us consider an

EXAMPLE. With $A := \{(x, y)^T \in \mathbb{R}^2 \mid x^2 \geqslant y \geqslant 0,\ x \geqslant 0\}$ define $f: A \to \mathbb{R}^2$ by

$$f\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} x \\ y + y^2/x^2 \end{pmatrix}, \qquad x \neq 0,$$

$$:= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad x = y = 0.$$

f is C^1 for all $z \in A$ in the sense of Definition 1 (including $z = (0, 0)^T$), f is even a C^1-*diffeomorphism*, that is, $f^{-1} : f(A) \to B_1$ exists and is also C^1. Nevertheless, there exists no *extension* of f to a $\mathbb{R}^2$-neighborhood of $(0, 0)^T$ such that the extended function is continuously Fréchet-differentiable at $(0, 0)^T$ (see [4]).

In the following, differentiability is always meant in the sense of Definition 1 *unless otherwise stated.*

Remark. Definition 1 allows us to speak of differentiability even for functions which are defined on domains with cusps, that is, where the dimension of the tangent cone of a point of the domain varies with that point. The differential topology of such manifolds with cusps (which include the set of γ-polynomials of degree n) is quite interesting and worthy of more investigations. Our parametrization p is defined on a polyhedral set and therefore some of the pathologies of the general case cannot occur. Namely, the derivative of p must be unique if it exists.

THEOREM 2. *For a $(k+1)$-kernel with $k \leqslant n$ and for arbitrary $\hat{a} = (\hat{a}_1,\dots,\hat{a}_n, \hat{\tau}_1, 0,\dots,0, \hat{\tau}_2, 0,\dots, \hat{\tau}_l, 0,\dots,0)^T \in \mathbb{R}^n \times M$ with $p(\hat{a}) \in \Gamma^{\gamma}_{n,k} \backslash \Gamma^{\gamma}_{n-1,k}$ the parametrization p is C^1 in $(\mathbb{R}^n \times M) \cap U$ for a neighborhood U of $\hat{a}$ and regular in $\hat{a}$ and for all parameters from $(\mathbb{R}^n \times M)$ with all characteristic numbers $t_j^{(i)}$ distinct.*

The following lemma allows us to prove Theorem 2 by analyzing partial derivatives only; see Krabs [9] for a similar lemma in the case of Fréchet-differentiable functions.

LEMMA 1. *For a polyhedron $C \subset \mathbb{R}^m$ (intersection of a finite number of half-spaces) let $G: C \times X \to \mathbb{R}$ be continuous with respect to both arguments. Assume that for arbitrary $c \in C$, $c + d \in C$, $x \in X$ the directional derivative $G'(c, x; d)$ (from c in direction d) exists and is continuous with respect to all three arguments. Then*

$$g: C \to (C(X), \|\cdot\|_q), \qquad g(c)(x) := G(c, x), \qquad \forall c \in C, x \in X,$$

is C^1 for all q, $1 \leqslant q \leqslant \infty$.

Remark. The lemma can easily be extended to cover the case of compact topological spaces X.

Proof of Lemma 1. For $c \in C$, $c + c_1,\dots, c + c_m \in C$, and $c_1,\dots, c_m$ linearly independent and normalized ($\|c_i\| = 1$) we define $g'(c)$ by

$$g'(c)\left(\sum_{i=1}^{m} \delta_i c_i\right)(x) := \sum_{i=1}^{m} \delta_i G'(c, x; c_i).$$

For $c + \Sigma \delta_i c_i \in C$ an application of the mean-value-theorem yields $\lambda_i \in [0, 1]$:

$$\begin{aligned} &g\left(c + \sum_{i=1}^{m} \delta_i c_i\right)(x) - g(c)(x) \\ &= \sum_{i=1}^{m} \left[g\left(c + \sum_{j=1}^{i} \delta_j c_j\right)(x) - g\left(c + \sum_{j=1}^{i-1} \delta_i c_i\right)(x)\right] \\ &= \sum_{i=1}^{m} \delta_i G'(c, x, c_i) \\ &\quad + \sum_{i=1}^{m} \delta_i \left[G'\left(c + \sum_{j=1}^{i-1} \delta_j c_j + \lambda_i \delta_i c_i, x; c_i\right) - G'(c, x; c_i)\right]. \end{aligned}$$

For a constant k we get, with $\delta := \|(\delta_1,\dots,\delta_m)^T\|$,

$$\left\| g\left(c+\sum_{i=1}^m \delta_i c_i\right) - g(c) - g'(c)\left(\sum_{i=1}^m \delta_i c_i\right)\right\|_q$$
$$\leqslant k\cdot \|(\delta_1,\dots,\delta_m)^T\| \cdot \sup_{\substack{x\in X, d\in C\\ d+c_i\in C\\ \|c-d\|<\delta}} \max_{1\leqslant i\leqslant m}$$
$$\times |G'(d,x;c_i)-G'(c,x;c_i)| \cdot \left(\int_X 1\,dx\right)^{1/q}.$$

The supremum tends to zero because X is compact and $G'(\cdot,\cdot;\cdot)$ is continuous. This proves differentiability. It is easily seen that g is indeed C^1. ∎

Proof of Theorem 2. In view of Lemma 1, to prove that p is continuously differentiable, we have only to investigate partial derivatives. As before we discuss only the case where all characteristic numbers coalesce ($l=1$, $k=n$) in order to avoid confusion by too many indices. In view of Lemma 1, it is sufficient to prove the following three convergence results for

$\mathbb{R}^n \times M \ni a = (a_1,\dots,a_n,\tau,\delta_1,\dots,\delta_{n-1})^T \to \hat{a} = (\hat{a}_1,\dots,\hat{a}_n,\hat{\tau},0,\dots,0)^T$:

$$\frac{\partial p}{\partial a_i} = \sum_{1\leqslant j_1<j_2<\cdots<j_i\leqslant n} \Delta_t^{i-1}(t_{j_1},\dots,t_{j_i})\,\gamma \to \binom{n}{i} \Delta_t^{i-1}(\hat{\tau},\dots,\hat{\tau})\,\gamma, \tag{1}$$

$$\frac{\partial p}{\partial \tau} = \sum_{i=1}^n a_i \sum_{1\leqslant j_1<\cdots<j_i\leqslant n} \sum_{k=1}^i \Delta_t^i(t_{j_1},\dots,t_{j_i},t_{j_k})\,\gamma$$
$$\to \sum_{i=1}^n \hat{a}_i \binom{n}{i}\cdot i\cdot \Delta_t^i(\hat{\tau},\dots,\hat{\tau})\,\gamma, \tag{2}$$

$$\frac{\partial p}{\partial \delta_i} = \sum_{j=1}^n a_j \sum_{1\leqslant j_1<\cdots<j_j\leqslant n} \frac{\partial}{\partial \delta_i} \Delta_t^{j-1}(t_{j_1},\dots,t_{j_j})\,\gamma$$
$$\to \sum_{j=2}^{n+2} b_j^i \Delta_t^{j-1}(\hat{\tau},\dots,\hat{\tau})\,\gamma \qquad \text{for certain } b_j^i\in\mathbb{R},$$
$$b_{n+2}^i\cdot \hat{a}_n > 0. \tag{3}$$

The proof of (1) and (2) is straightforward. To prove (3) we analyze

$$\frac{\partial}{\partial \delta_{n-1}} \sum_{1\leqslant j_1<\cdots<j_i\leqslant n} \Delta_t^{i-1}(t_{j_1},\dots,t_{j_i})\,\gamma, \qquad 1\leqslant i\leqslant n. \tag{4}$$

Partial derivatives with respect to δ_i $(1 \leqslant i < n-1)$ can be derived with the help of and in analogy to the analysis of (4). For $1 \leqslant i \leqslant n$ we have

$$\begin{aligned} 2\sqrt{\delta_{n-1}} \cdot \frac{\partial}{\partial \delta_{n-1}} \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant n} & \Delta_t^{i-1}(t_{j_1}, \ldots, t_{j_i})\, \gamma \\ = - \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant n} & \sum_{\substack{k=1 \\ j_k \neq n}}^{i} \Delta_t^i(t_{j_1}, \ldots, t_{j_i}, t_{j_k})\, \gamma \\ + \sum_{1 \leqslant j_1 < \cdots < j_{i-1} < n} & (n-1)\, \Delta_t^i(t_{j_1}, \ldots, t_{j_{i-1}}, t_n, t_n)\, \gamma, \end{aligned}$$

where the differentiation has been performed; next, we reorder the difference quotients of order i which then enables us to rewrite the expression as a sum of difference quotients of order $i+1$.

$$\begin{aligned} = \sum_{1 \leqslant j_1 < \cdots < j_{i-1} < n} & \left\{ \sum_{\substack{k=1 \\ k \neq j_1 \cdots k \neq j_{i-1}}}^{n-1} [\Delta_t^i(t_{j_1}, \ldots, t_{j_{i-1}}, t_n, t_n)\, \gamma - \Delta_t^i(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_k)\, \gamma] \right. \\ & \left. + \sum_{k=1}^{i-1} [\Delta_t^i(t_{j_1}, \ldots, t_{j_{i-1}}, t_n, t_n)\, \gamma - \Delta_t^i(t_{j_1}, \ldots, t_{j_{i-1}}, t_n, t_{j_k})\, \gamma] \right\} \\ = \sum_{k=1}^{n-1} (t_n - t_k) & \left\{ \sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-1} < n \\ k \neq j_1, \ldots, k \neq j_{i-1}}} [\Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_n, t_n)\, \gamma + \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_k, t_n)\, \gamma] \right. \\ & \left. + \sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-2} < n \\ k \neq j_1, \ldots, k \neq j_{i-2}}} \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-2}}, t_k, t_k, t_n, t_n)\, \gamma \right\} \\ = \sqrt{\delta_{n-1}} \cdot n \cdot \sum_{k=1}^{n-1} & \left(\sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-1} < n \\ k \neq j_1, \ldots, k \neq j_{i-1}}} [\Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_n, t_n)\, \gamma + \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_k, t_n)\, \gamma] \right. \\ & \left. + \sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-2} < n \\ k \neq j_1, \ldots, k \neq j_{i-2}}} \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-2}}, t_k, t_k, t_n, t_n)\, \gamma \right) \\ + \sum_{j=1}^{n-2} \sqrt{\delta_j} \cdot & \left\{ \sum_{k=1}^{j} \left(\sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-1} < n \\ k \neq j_1, \ldots, k \neq j_{i-1}}} [\Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_n, t_n)\, \gamma + \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-1}}, t_k, t_k, t_n)\, \gamma] \right. \right. \\ & \left. \left. + \sum_{\substack{1 \leqslant j_1 < \cdots < j_{i-2} < n \\ k \neq j_1, \ldots, k \neq j_{i-2}}} \Delta_t^{i+1}(t_{j_1}, \ldots, t_{j_{i-2}}, t_k, t_k, t_n, t_n)\, \gamma \right) \right. \end{aligned}$$

$$-j\cdot \sum_{\substack{1\leqslant j_1<\cdots<j_{i-1}<n\\ j+1\neq j_1,\ldots,j+1\neq j_{i-1}}} [\Delta_t^{i+1}(t_{j_1},\ldots,t_{j_{i-1}},t_{j+1},t_n,t_n)\,\gamma + \Delta_t^{i+1}(t_{j_1},\ldots,t_{j_{i-1}},t_{j+1},t_{j+1},t_n)\,\gamma]$$

$$-j\cdot \sum_{\substack{1\leqslant j_1<\cdots<j_{i-2}<n\\ j+1\neq j_1,\ldots,j+1\neq j_{i-2}}} \Delta_t^{i+1}(t_{j_1},\ldots,t_{j_{i-2}},t_{j+1},t_{j+1},t_n,t_n)\,\gamma\Big\}.$$

The last transformation reorders the difference quotients according to their factors $\sqrt{\delta_j}$ and requires some detailed bookkeeping.

Since δ_j/δ_{n-1} is bounded (for arguments of p(!!)) and since the last expression enclosed in braces converges to zero for $\delta_{n-1}\to 0$, we get

$$\frac{\partial}{\partial\delta_{n-1}} \sum_{1\leqslant j_1<\cdots<j_i\leqslant n} \Delta_t^{i-1}(t_{j_1},\ldots,t_{j_i})\,\gamma \tag{5}$$
$$\to \frac{n}{2}(n-1)\left[\binom{n-2}{i-2}+2\cdot\binom{n-2}{i-1}\right]\Delta_t^{i+1}(\hat{\tau},\ldots,\hat{\tau})\,\gamma \quad (1\leqslant i\leqslant n).$$

(1), (2), and (3) allow us to apply Lemma 1, which proves the claimed differentiability.

Next, we prove the regularity of p: Let $(\Delta a_1,\ldots,\Delta a_n,\Delta\tau,\Delta\delta_1,\ldots,\Delta\delta_{n-1})$ be a feasible perturbation of $\hat{a}$ with at least one $\Delta\delta_i$ nonzero. For certain $h_1,\ldots,h_{n+2}\in\mathbb{R}$ we have

$$\sum_{i=1}^{n}\Delta a_i\frac{\partial p}{\partial a_i}+\Delta\tau\frac{\partial p}{\partial\tau}+\sum_{i=1}^{n-1}\Delta\delta_i\frac{\partial p}{\partial\delta_i}$$
$$=\sum_{i=1}^{n+2}h_i\Delta_t^{i-1}(\tau,\ldots,\tau)\,\gamma.$$

Only the $\Delta\delta_i\,(\partial p/\partial\delta_i)$ conribute to h_{n+2} and the contribution is zero or has the same sign as $\hat{a}$ which is nonzero because $p(\hat{a})$ has maximal degree, $p(\hat{a})\in\Gamma_{n,k}^{\gamma}\backslash\Gamma_{n-1,k}^{\gamma}$. Thus, h_{n+2} is nonzero iff at least one $\Delta\delta_i$ is nonzero. This proves the regularity for a perturbation with at least one $\Delta\delta_i$ nonzero. If all $\Delta\delta_i$ are zero the regularity follows from the linear independence of $\{\Delta^0(\tau)\,\gamma,\ldots,\Delta^n(\tau,\ldots,\tau)\,\gamma\}$. Therefore, p is regular in $\hat{a}$.

The regularity in interior points (all characteristic numbers distinct) can be reduced to the regularity of the parametrization $\sum_{i=1}^{n}a_i\Delta_t^{i-1}(t_1,\ldots,t_n)\,\gamma$ for all characteristic numbers distinct. ∎

4. The Special Case $n = 3$

To better illustrate the ideas of the preceding two sections let us consider the case $n = 3$. The classical parametrization

$$\left\{ \sum_{i=1}^{3} a_i \gamma(t_i, \cdot) \middle| a_i, t_i \in \mathbb{R}, t_1 \leqslant t_2 \leqslant t_3 \right\} \tag{6}$$

does not even allow one to represent elements in the closure: For example,

$$\frac{\gamma(t+\delta, \cdot) - \gamma(t, \cdot)}{\delta}$$

is an element of (6) for each $\delta > 0$, but the limit $(\partial/\partial t)\, \gamma(t, \cdot)$ for $\delta \to 0$ is not in (6). To overcome this difficulty Werner and Braess [2] used difference quotients to parameterize γ-polynomials:

$$\left\{ \sum_{i=1}^{3} a_i \Delta_t^{i-1}(t_1, \ldots, t_i)\, \gamma(t, \cdot) \middle| a_i, t_i \in \mathbb{R}, t_1 \leqslant t_2 \leqslant t_3 \right\}.$$

This induces indeed a homeomorphic representation of $\Gamma^n \backslash \Gamma^{n-1}$ and is therefore of great help for all problems concerning the topological structure of γ-polynomials. But for coalescing characteristic numbers ($t_1 = t_2 = t_3 = \tau$) the partial derivatives of Braess' parametrization span only a four-dimensional space, whereas our parametrization p spans the full tangent cone, a five-dimensional space with a sign-restriction.

Let us reconsider our parametrization for the special case $n = 3$ for three coalescing characteristic numbers (that is, $l = 1$, $m_1 = 3$). This gives us

$$M = \{(\tau, \delta_1, \delta_2)^T \in \mathbb{R}^3 \,|\, 0 \leqslant \delta_1 \leqslant 9 \cdot \delta_2\},$$

$$\begin{aligned} &p((a_1, a_2, a_3, \tau, \delta_1, \delta_2)^T) \\ &\quad = a_1[\Delta_t^0(t_1)\, \gamma + \Delta_t^0(t_2)\, \gamma + \Delta_t^0(t_3)\, \gamma] \\ &\qquad + a_2[\Delta_t^1(t_1, t_2)\, \gamma + \Delta_t^1(t_1, t_3)\, \gamma + \Delta_t^1(t_2, t_3)\, \gamma] \\ &\qquad + a_3 \Delta_t^2(t_1, t_2, t_3)\, \gamma \end{aligned}$$

with

$$t_1 := \tau - \sqrt{\delta_1} - \sqrt{\delta_2}; \quad t_2 := \tau + \sqrt{\delta_1} - \sqrt{\delta_2}; \quad t_3 := \tau + 2\sqrt{\delta_2}$$

and $a_3 \neq 0$ if $p(\cdot) \in \Gamma_3^\gamma \backslash \Gamma_2^\gamma$ and $t_3 - t_1$ are sufficiently small.

From Theorem 1 we conclude $p(\mathbb{R}^3 \times M) = \Gamma_3^\gamma$, which is indeed an immediate consequence of the identity

$$\sum_{i=1}^{3} a_i \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant 3} \Delta_t^{i-1}(t_{j_1}, \dots, t_{j_i})\,\gamma$$

$$= 3a_1 \Delta_t^0(t_1)\,\gamma + [3a_2 + a_1(t_2 - t_1 + t_3 - t_1)]\,\Delta_t^1(t_1, t_2)\,\gamma$$

$$+ [a_3 + a_1(t_3 - t_1) + a_2(t_3 - t_2) + a_2(t_3 - t_1)]\,\Delta_t^2(t_1, t_2, t_3)\,\gamma.$$

The differentiability of p follows from the continuity of the partial derivatives with Lemma 1. For a parameter from $\mathbb{R}^2 \times M$ with $\delta_2 \neq 0$ we get, for the partial derivative with respect to δ_2, after some intermediate calculations,

$$\frac{\partial p}{\partial \delta_2} = \frac{3}{2} a_1 [\Delta_t^2(t_1, t_1, t_3)\,\gamma + \Delta_t^2(t_1, t_3, t_3)\,\gamma$$

$$+ \Delta_t^2(t_2, t_2, t_3)\,\gamma + \Delta_t^2(t_2, t_3, t_3)\,\gamma]$$

$$+ \frac{3}{2} a_2 [\Delta_t^3(t_1, t_1, t_2, t_3)\,\gamma + \Delta_t^3(t_1, t_2, t_2, t_3)\,\gamma$$

$$+ 4\Delta_t^3(t_1, t_2, t_3, t_3)\,\gamma]$$

$$+ \frac{3}{2} a_3 [\Delta_t^4(t_1, t_1, t_2, t_3, t_3)\,\gamma + \Delta_t^4(t_1, t_2, t_2, t_3, t_3)\,\gamma]$$

$$+ \frac{a_1}{2} \sqrt{\frac{\delta_1}{\delta_2}}\, [\Delta_t^2(t_1, t_1, t_3)\,\gamma + \Delta_t^2(t_1, t_3, t_3)\,\gamma$$

$$- \Delta_t^2(t_2, t_2, t_3)\,\gamma - \Delta_t^2(t_2, t_3, t_3)\,\gamma]$$

$$+ \frac{3a_2}{2} \sqrt{\frac{\delta_1}{\delta_2}}\, [\Delta_t^3(t_1, t_1, t_2, t_3)\,\gamma - \Delta_t^3(t_1, t_2, t_2, t_3)\,\gamma]$$

$$+ \frac{a_3}{2} \sqrt{\frac{\delta_1}{\delta_2}}\, [\Delta_t^4(t_1, t_1, t_2, t_3, t_3)\,\gamma - \Delta_t^4(t_1, t_2, t_2, t_3, t_3)\,\gamma].$$

$\partial p / \partial \delta_2$ is continuous for $\delta_2 \to 0$ *only* because $0 \leqslant \sqrt{\delta_1} \leqslant 3\sqrt{\delta_2}$ (or $\sqrt{\delta_1/\delta_2}$ is bounded) for arguments from $\mathbb{R}^3 \times M(!)$.

The derivatives for three coalescing characteristic numbers ($\delta_1 = \delta_2 = 0$) are

$$\frac{\partial p}{\partial a_1} = 3\Delta_t^0(\tau)\,\gamma; \frac{\partial p}{\partial a_2} = 3\Delta_t^1(\tau, \tau)\,\gamma; \frac{\partial p}{\partial a_3} = \Delta_t^2(\tau, \tau, \tau)\,\gamma;$$

$$\frac{\partial p}{\partial \tau} = 3a_1\Delta_t^1(\tau, \tau)\,\gamma + 6a_2\Delta_t^2(\tau, \tau, \tau)\,\gamma + 3a_3\Delta_t^3(\tau, \tau, \tau, \tau)\,\gamma;$$

$$\frac{\partial p}{\partial \delta_1} = 2a_1\Delta_t^2(\tau, \tau, \tau)\,\gamma + 3a_2\Delta_t^3(\tau, \tau, \tau, \tau)\,\gamma + a_3\Delta_t^4(\tau, \tau, \tau, \tau, \tau)\,\gamma;$$

$$\frac{\partial p}{\partial \delta_2} = 6a_1\Delta_t^2(\tau, \tau, \tau)\,\gamma + 9a_2\Delta_t^3(\tau, \tau, \tau, \tau)\,\gamma + 3a_3\Delta_t^4(\tau, \tau, \tau, \tau, \tau)\,\gamma.$$

The derivative is *regular* because only *nonnegative* perturbations are feasible for the vanishing $\delta_1 = \delta_2 = 0$. Thus, there is no perturbation h with $h + (a_1, a_2, a_3, \tau, 0, 0)^T \in \mathbb{R}^3 \times M$ and $h_1(\partial p/\partial a_1) + \cdots + h_6(\partial p/\partial \delta_2) \equiv 0$. Once regularity is established it is easy to calculate the tangent cones. The tangent cone at $p((a_1, a_2, a_3, \tau, 0, 0)^T)$ with $a_3 \neq 0$ is

$$\left\{ \sum_{i=1}^{3} h_i \frac{\partial p}{\partial a_i} + h_4 \frac{\partial p}{\partial \tau} + h_5 \frac{\partial p}{\partial \delta_1} + h_6 \frac{\partial p}{\partial \delta_2} \,\middle|\, h_i \in \mathbb{R}, 0 \leqslant 0 + h_5 \leqslant 9(0 + h_2) \right\}$$

$$= \left\{ \sum_{i=1}^{5} h_i' \Delta_t^{i-1}(\tau, ..., \tau)\,\gamma(t, \cdot) \,\middle|\, h_i' \in \mathbb{R}, \operatorname{sgn}(a_3 \cdot h_5) \geqslant 0 \right\}.$$

The parametrization is not necessarily regular, though, for two coalescing characteristic numbers. For $a := (0, 0, a_3, \tau, \delta_1, 9\delta_1)^T \in \mathbb{R}^3 \times M$ and $\delta_1 > 0$ we have

$$3\frac{\partial}{\partial \delta_1} p(a) - \frac{\partial}{\partial \delta_2} p(a) = 0 \in C(X).$$

If the tangent cone of a γ-polynomial is to be calculated the polynomial has to be fixed first and then a parametrization can be constructed which is regular at that point.

5. Concluding Remarks

In the preceding sections we discussed a novel parametrization p for γ-polynomials which has two advantages over older parametrizations: (i) It is differentiable and regular even for coalescing characteristic numbers and (ii) the parameter sets are simply polyhedral. This makes it very easy to calculate the tangent cone of an element $g \in \Gamma_n \backslash \Gamma_{n-1}$ for a normal γ-polynomial: The tangent cone is simply the cone spanned by the image of all feasible perturbations under the derivative of p at that point. Thus, our parametrization helps establish invariant properties of γ-polynomials, that is, properties which are independent of individual parametrizations. This is a big step towards another invariant property: In a subsequent paper [5] we rely heavily on tangent cones to establish necessary and sufficient conditions for a γ-polynomial to be a local best approximation.

REFERENCES

1. C. DE BOOR, On the approximation by γ-polynomials, *in* "Approximations with Special Emphasis on Spline-Functions" (I. J. Schoenberg, Ed.), Academic Press, New York London, 1969.
2. D. BRAESS, Über die Vorzeichenstruktur der Exponentionalsummen, *J. Approx. Theory* **3** (1970), 101–113.
3. D. BRAESS, Chebyshev approximation by γ-polynomials, I, II, III, *J. Approx. Theory* **9** (1973), 20–43; **11** (1974), 16–37; **24** (1978) 119–145.
4. L. J. CROMME, Approximation auf Mannigfaltigkeiten mit Spitzen-Theorie und numerische Methoden, Habilitationsschrift, Göttingen, 1979.
5. L. J. CROMME, A unified approach to differential characterizations of local best approximations for exponential sums and splines, submitted.
6. L. J. CROMME, Algorithms for approximation by exponential sums and splines, in preparation.
7. F. B. HILDEBRAND, "Introduction to Numerical Analysis," McGraw–Hill, New York, 1956.
8. C. R. HOBBY AND J. R. RICE, Approximation from a curve of Functions, *Arch. Rat. Mech. Anal.* **24** (1967), 91–106.
9. W. KRABS, Über differenzierbare asymptotisch konvexe Funktionenfamilien bei der nichtlinearen gleichmäßigen Approximation, *Arch. Rat. Mech. Anal.* **27** (1967), 275–288.